UNIVERSITY OF SWAZILAND

FINAL EXAMINATION PAPER 2017

TITLE OF PAPER	:	DISTRIBUTION THEORY
COURSE CODE	:	ST301
TIME ALLOWED	:	TWO (2) HOURS
REQUIREMENTS	:	CALCULATOR
INSTRUCTIONS	:	ANSWER ANY THREE QUESTIONS

Question 1 **[20 marks, 10+10]**

(a) Let X and Y be independent, uniform random variables on $[0, 1]$. Find the density function and distribution function for $X + Y$.

(b) Suppose that random variable X has mgf M_X given by

$$M_X(t) = \frac{1}{8}e^t + \frac{2}{8}e^{2t} + \frac{5}{8}e^{3t}.$$

Find the probability distribution, and the expectation and variance of X.

Question 2 **[20 marks, 6+2+4+8]**

(a) A markov chain Z_t, $t = 0, 1, 2, \cdots$ on the state space $\mathfrak{S} = \{1, 2, 3, 4, 5\}$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0.3 & 0.4 & 0.3 & 0 & 0 \\ 0 & 0 & 0.3 & 0.4 & 0.3 \\ 0 & 0.3 & 0 & 0.3 & 0.4 \\ 0 & 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0 & 0.6 & 0.4 \end{pmatrix} \end{array}.$$

Classify all states. Which states are recurrent and which states are transient? Find all closed classes. Is the chain *ergodic*?

(b) Customers arrive at a bank according to a Poisson process with rate 6 customers per hour.

(i) What is the probability that five customers arrive in the first hour after the bank opens?

(ii) What is the probability that five customers arrive in the first hour after the bank opens and that three customers arrive in the second hour after the bank opens?

(iii) Suppose it is known that five customers arrived in the first hour after the bank opened. What is the probability that exactly one of the five customers arrived during the first 20 minutes? Justify your answer using a theorem.

Question 3 **[20 marks, 2+4+3+3+8]**

(a) State what is meant by the memoryless property of the exponential distribution.

(b) Prove the memoryless property for the exponential distribution.

(c) Let $\mathbf{G}$ be the generator of a continuous time Markov chain $X(t)$, with $t \geq 0$, and let $\mathbf{P}(t)$ be the matrix such that $\mathbb{P}(t)_{i,j} = \mathbb{P}(X(s+t) = j | X(s) = i)$.

(i) State the equation for $\mathbf{P}(t)$ in terms of $\mathbf{G}$.

(ii) State the backwards and forwards Kolmogorov equations.

(d) A markov chain Z_t, $t = 0, 1, 2, \cdots$ on the state space $\mathfrak{S} = \{0, 1, 2, 3, 4\}$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Determine the mean time to reach state 4 starting from state 1 using a first step analysis.

Question 4 [20 marks, 8+6+6]

(a) Suppose $N \sim$ Geometric(p) and each $X_i \sim$ Binomial(1,θ) (independent). Find the distribution of $Z = \sum_{i=1}^{N} x_i$.

(b) Suppose that X and Y are continuous random variables with joint pdf given by

$$f)X, Y(x, y) = \frac{1}{2x^2 y}, \qquad 1 \leq x < \infty, \frac{1}{x} \leq y < x,$$

and zero otherwise.

Derive

(i) the **marginal** pdf of Y,

(ii) the **conditional** pdf of X given $Y = y$.

Question 5 [20 marks, 8+3+4+5]

(a) We agree to try to meet between 12 and 1 for lunch at our favorite sandwich shop. Because of our busy schedules, neither of us is sure when we'll arrive; we assume that for each of us our arrival time is uniformly distributed over the hour. So that neither of us has to wait too long, we agree that we will each wait exactly 15 minutes for the other to arrive, and then leave.

What is the probability we actually meet each other for lunch?

(b) Without a vaccine, the death rate is 0.1 for a patient with disease D_1, and 0.5 for a patient with disease D_2. If a patient receives vaccine A and still develops a disease, the death rate is 0.06 for disease D_1 and 0.1 for disease D_2. If a patient receives vaccine B and still develops a disease, the corresponding death rates are 0.01 and 0.2, respectively.

(i) Show that for any events A, B and C, we have

$$P(A \cap B \cap C) = P(C|A \cap B)P(B \cap A|A)P(A)$$

(ii) Find the probability that a particular individual satisfies the conditions that he/she is not vaccinated, develops disease D_2, and dies eventually.

(iii) Given that a patient developed disease D_2 and died, find the probability that the patient has not been vaccinated.

Question 6 [20 marks, 8+6+6]

(a) A markov chain Z_t, $t = 0, 1, 2, \cdots$ on the state space $\mathfrak{S} = \{1, 2, 3, 4\}$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0.2 & 0.2 & 0.3 & 0.3 \\ 0.5 & 0 & 0.5 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \end{pmatrix} \end{array}$$

with the three-step transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0.383 & 0.243 & 0.287 & 0.087 \\ 0.320 & 0.220 & 0.355 & 0.105 \\ 0.320 & 0.355 & 0.230 & 0.105 \\ 0.350 & 0.225 & 0.275 & 0.150 \end{pmatrix} \end{array}.$$

If the initial probability distribution is

$$\{0.3, 0.3, 0.2, 0.2\}$$

what is $\mathbb{P}(Z_3 = 2, Z_6 = 3, Z_9 = 4)$?

(b) A small barbershop is operated by two barbers, each of which has a chair for a single customer. When two customers are in the store, its doors are locked so no more customers can enter. The remaining time, when there are fewer than two customers in the store, its doors are left open and potential customers arrive according to a Poisson process with rate of three per hour. Successive service times are independent exponential random variables with expectation of $\frac{1}{4}$ hour.

(i) Find the generator for $X(t)$, the number of customers in the shop.

(ii) Find the limiting probabilies $\pi_i = \lim_{t \to \infty} \mathbb{P}(X(t) = i)$ for $i = 0, 1, 2$.