UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION PAPER 2017/2018

TITLE OF PAPER	:	DISTRIBUTION THEORY
COURSE CODE	:	ST301
TIME ALLOWED	:	TWO (2) HOURS
REQUIREMENTS	:	CALCULATOR
INSTRUCTIONS	:	ANSWER ANY THREE QUESTIONS

Question 1 [20 marks, 4+6+10]

(a) Components in a machine fail and are replaced according to a Poisson process of rate 4 a month.

(i) Find the probability that exactly 4 fail in the first month and exactly 8 fail in the first two months.

(ii) Find the probability that at least 2 fail in the first month and at least 4 fail in the first 2 months.

(b) A population starts off with just one female. Each female lives for a fixed time T and then dies. The number of offspring of a single female is given by the following probability distribution:

m	0	1	2
p_m	0.3	0.4	0.3

Different female are independent. Let $G_n(s)$ denote the probability generating function of the number of females at the nth generation and write $G_1(s) = G(s)$. Write down the polynomial expression for $G(s)$. Working from first principles use a first step argument to show that

$$G_2(s) = G(G(s)).$$

Hence deduce the probability distribution of the females in the second generation.

Question 2 [20 marks,10+4+6]

(a) Let $N \sim$ Poisson(μ). Define the random variable

$$Y = \begin{cases} X_1 \sim \text{Exponential}(\lambda) & N > 0; \\ X_2 \sim \text{Exponential}(2\lambda) & N = 0, \end{cases}$$

where N, X_1 and X_2 are independent of each other. Derive the moment generating function of Y, and find $E(Y)$ and Var(Y).

(b) Let X and Y be two independent standard normal random variables. Consider random variables U and V such that X and Y can be represented by

$$\begin{cases} X = U\cos V, \\ Y = U\sin V. \end{cases}$$

You are given some useful properties of sin and cos functions:

$$\frac{d}{dx}(\sin x) = \cos x, \quad \frac{d}{dx}(\cos x) = -\sin x, \quad \sin^2(x) + \cos^2(x) = 1.$$

Also, over $[0, 2\pi)$, $\sin \geq 0$ for $x \in [0, \pi]$ and $\cos x \geq 0$ for $xin[0, \pi/2]$ of $[3\pi/2, 2\pi)$,

(i) Give the respective ranges for U and V in order that the transformation defined is one to one. With this, find U and V in terms of X and Y.

(ii) Find the joint probability density of $f_{U,V}(u, v)$ of U and V.

Question 3 [20 marks, 6+8+6]

(a) A markov chain Z_t, $t = 0, 1, 2, \cdots$ on the state space $\mathfrak{S} = \{1, 2, 3, 4, 5\}$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0 & 0.3 & 0.3 & 0.2 & 0.2 \\ 0 & 0 & 0.3 & 0.3 & 0.4 \\ 0 & 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0 & 0.6 & 0.4 \end{pmatrix} \end{array}$$

with the two-step transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0.04 & 0.10 & 0.1*6 & 0.34 & 0.36 \\ 0 & 0.09 & 0.18 & 0.35 & 0.38 \\ 0 & 0 & 0.09 & 0.45 & 0.46 \\ 0 & 0 & 0 & 0.52 & 0.48 \\ 0 & 0 & 0 & 0.48 & 0.52 \end{pmatrix} \end{array}.$$

(i) If the initial probability distribution is

$$\{0.2, 0.2, 0.2, 0.3, 0.1\}$$

what is $\mathbb{P}(Z_2 = 4 \text{ or } 5)$?

(ii) Determine the mean time to reach state 4 or 5 starting from state 1 using a first step analysis.

(b) Let Z_t, $t = 0, 1, 2, \cdots$ be a Markov chain on a finite state space $\mathfrak{S} = \{1, 2, \cdots, n\}$. State the Markov property clearly. Define the 1-step transition probabilities

$$p_{ij} = \mathbb{P}(Z_{t+1} = j | Z_t = i) \qquad i, j \in \mathfrak{S}, \quad t = 0, 1, 2, \cdots.$$

Using the Markov property, show

$$\mathbb{P}(Z_2 = j | Z_0 = i) = \sum_{k=1}^{n} p_{ik} p_{kj}$$

where $i, j \in \mathfrak{S}$.

Question 4 [20 marks, 8+2+4+6]

(a) A markov chain Z_t, $t = 0, 1, 2, \cdots$ on the state space $\mathfrak{S} = \{1, 2, 3\}$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.6 & 0 & 0.4 \\ 0.3 & 0.7 & 0 \end{pmatrix} \end{array}.$$

Explain why the stationary distribution exists. Find the stationary probability distribution.

(b) A markov chain Z_t, $t = 0, 1, 2, \cdots$ on the state space $\mathfrak{S} = \{1, 2, 3, 4, 5\}$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 & 0 \\ 0.7 & 0 & 0 & 0.3 & 0 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

(i) Which states are absorbing?

(ii) Find the probability of absorption occurs at state 5 starting from state 3. (You need to set up the difference equations by first step analysis and solve them.)

(c) Let Z_t, $t = 0, 1, 2, \cdots$ be a Markov chain on a finite state space $\mathfrak{S} = \{1, 2, \cdots, n\}$. State the Markov property clearly. Define the 1-step transition probabilities

$$p_{ij} = \mathbb{P}(Z_{t+1} = j | Z_t = i) \qquad i, j \in \mathfrak{S}, \quad t = 0, 1, 2, \cdots.$$

Using the Markov property, show

$$\mathbb{P}(Z_2 = j | Z_0 = i) = \sum_{k=1}^{n} p_{ik} p_{kj}$$

where $i, j \in \mathfrak{S}$.

Question 5 [20 marks, 3+9+8]

Let X and Y be random variables with joint density

$$f_{X,Y}(x, y) = \begin{cases} ke^{-\lambda x}, & 0 < y < x < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find k.

(b) Derive the marginal density for Y and hence evaluate $E(Y)$, $E(Y^2)$ and $Var(Y)$.

(c) Derive the conditional density, $f_{X|Y}(x|y)$, and the conditional expectation, $E[X|Y]$. Hence or otherwise, evaluate $E(X)$ and $Cov(X, Y)$.

Question 6 [20 marks, 4+5+2+5+4]

(a) The number of claims received at an insurance company during a week is a random variable with mean 20 and variance 120. The amount paid in each claim is a random variable with mean 350 and variance 10000. Assume that the amounts of different claims are independent.

(i) Suppose this company received exactly 3 claims in a particular week. The amount of each claim is still random as already specified. What are the mean and variance of the total amount paid to these 3 claims in this week?

(ii) Assume that in one week, all claims received the same payment of 300. What is the mean and variance of the total amount paid in this week?

(b) Let Z be a random variable with density

$$f_Z(z) = \frac{1}{2}e^{-|z|}, \qquad \text{for } 0 < z < \infty.$$

(i) Show that f_Z is a valid density.

(ii) Find the moment generating function of Z and specify the interval where the MGF is well-defined.

(iii) By considering the cumulant generating function or otherwise, evaluate $E(Z)$ and $Var(Z)$.