UNIVERSITY OF SWAZILAND

UMCULU SISEKELO SESIVE

MAIN EXAMINATION PAPER 2018

TITLE OF PAPER : PROBABILITY THEORY

COURSE CODE : STA 201

TIME ALLOWED : 3 HOURS

INSTRUCTIONS : ANSWER ANY FIVE QUESTIONS.

REQUIREMENTS : SCIENTIFIC CALCULATOR

Question 1

a) If P(A) = 0.25 and P(B) = 0.8, the show that $0.05 \leq P(A \cap B) \leq 0.25$.

(5 Marks)

b) Let A and B be Events in a sample space Ω such that $P(A) = \frac{1}{2} = P(B)$ and $P(A^c \cap B^c) = \frac{1}{3}$. Find $P(A \cup B^c)$.

(5 Marks)

c) A box of fuses contains 20 fuses, of which 5 are defective. If 3 of the fuses are selected at random and removed from the box in succession without replacement, what is the probability that all three fuses are defective?

(5 Marks)

d) Suppose box A contains 4 red and 5 blue chips and box B contains 6 red and 3 blue chips. A chip is chosen at random from the box A and placed in box B. Finally, a chip is chosen at random from among those now in box B. What is the probability a blue chip was transferred from box A to box B given that the chip chosen from box B is red?

(5 Marks)

Question 2

A continuous random variable X has cumulative distribution function:

$$F_X(x) = \begin{cases} 0, if\ x \leq 0 \\ \sqrt{x}, if\ 0 < x \leq 1 \\ 1, if\ x > 1 \end{cases}$$

a) Find the probability density function of X.

(4 Marks)

b) Calculate the expectation and variance of X.

(12 Marks)

c) Calculate the lower quartile of X.

(4 Marks)

Question 3

a) The random variable X is uniformly distributed on the interval (0, 1). Derive the pdf of the random variable Y = −lnX , E(X), and Var(X).

(8 + 4+ 8 Marks)

Question 4

The joint probability mass function of the lifetimes X and Y of two connected components in a machine can be modelled by $p(x,y) = \frac{e^{-2}}{x!(y-x)!}$ for x = 0, 1,.... and $y = x, x+1, \ldots$.

a) What is the marginal distributions of X and Y?
b) What is the joint probability mass function of X and Y – X? Are X and Y – x independent?
c) What is the correlation between X and Y?

(7 + 7 + 6 Marks)

Question 5

The continuous random variables X and Y satisfy $f(y|x) = \frac{1}{x}$ for 0 < y < x, and $f(y|x) = 0$ otherwise. The marginal density function of X is given by $f_X(x) = 2x$ for $0 < x < 1$ and $f_X(x) = 0$ otherwise.

a) What is the conditional density? $f(x|y)$?
b) What is $E(X|Y = y)$?

(10 + 10 Marks)

Question 6

a) Let $X \sim N(\mu_1, \sigma_1^2)$, and $Y \sim N(\mu_2, \sigma_2^2)$. Furthermore X and Y are independent to each other. Derive the distribution of 2X - 3Y – 5.

b) A company insures homes in three cities J, K, L. The losses occurring in these cities are independent. The moment generating functions for the loss distributions of the cities are

$$M_J(t) = (1 - 2t)^{-3}, \quad M_K(t) = (1 - 2t)^{-2.5}, \quad M_L(t) = (1 - 2t)^{-4.5}$$

Let X represent the combined losses from the three cities. Calculate $E(X^3)$.

(10 + 10 Marks)

Question 7

a) An insurance policy pays a total medical benefit consisting of a part paid to the surgeon, X, and a part paid to the hospital, Y, so that the total benefit is X+Y. Suppose that Var(X) = E5, 000, Var(Y) = E10,000, and Var(X + Y) = E17, 000.

If X is the increased by a flat amount of E100, and Y is increased by 10%, what is the variance of the total benefit after these increases?

b) Given that E(X) = 5, $E(X^2)$ = 27.4, E(Y) = 7, E(Y2) = 51.4, and Var(X + Y) =8, find Cov(X + Y, X + 1.2Y).

(10 + 10 Marks)

Question 8

a) A diagnostic test for the presence of disease has two possible outcomes: 1 for disease present and 0 for disease not present. Let X denote the disease state of a patient, and let Y denote the outcome of a diagnostic test. The joint probability function of X and Y is given by:

$$P(X=x,Y=y)=\begin{cases}0.800 \ for\ (x,y)=(0,0)\\ 0.050 \ for\ (x,y)=(1,0)\\ 0.025 \ for\ (x,y)=(0,1)\\ 0.125 \ for\ (x,y)=(1,1)\end{cases}$$

Calculate Var(Y|X=1).

b) The stock prices of two companies at the end of any given year are modeles with random variables X and Y that follow a distribution with joint density function

$$f(x,y)=\begin{cases}2x & for\ 0<x<1, x<y<x+1\\ 0 & otherwise\end{cases}$$

What is the conditional variance of Y given that X = x?

(8 + 12 Marks)