UNIVERSITY OF SWAZILAND

FINAL EXAMINATION PAPER 2017

TITLE OF PAPER	:	MATHEMATICS FOR STATISTICIANS
COURSE CODE	:	STA213
TIME ALLOWED	:	TWO (2) HOURS
REQUIREMENTS	:	CALCULATOR
INSTRUCTIONS	:	THIS PAPER HAS FIVE (5) QUESTIONS. ANSWER ANY THREE (3) QUESTIONS.

Question 1 [20 marks, 4+4+4+4+1+1+1+1]

(a) Calculate

$$\int_0^1 \frac{4x+2}{1+x+x^2}\mathrm{d}x$$

, leaving your answer in the form $a \log b$, where a and b are integers.

(b) The 3×3 matrices $\mathbb{P}$ and $\mathbb{Q}$ satisfy

$$\mathbb{PQ} = \begin{pmatrix} k & 8 & 1 \\ 1 & 1 & 0 \\ 1 & 4 & 0 \end{pmatrix}, \quad \text{where}\mathbb{P} = \begin{pmatrix} k & 6 & 8 \\ 0 & 1 & 2 \\ -3 & 4 & 8 \end{pmatrix}$$

(i) Show that $\mathbb{PQ}$ is non-singular.

(ii) Find $(\mathbb{PQ})^{-1}$ in terms of k.

(iii) Find $\mathbb{Q}^{-1}$.

(c) Consider the vectors $\vec{v} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and compute the following quantities (showing all workings):

(i) $\vec{v} + \vec{w}$

(ii) $|\vec{v}|$

(iii) $\langle \vec{v}, \vec{w} \rangle$

(iv) Are $\vec{v}$ and $\vec{w}$ perpendicular? (Explain your answer).

Question 2 [20 marks, 3+3+6+4+4]

(a) Given a system of linear equations over $\mathbb{R}$

$$\begin{aligned} x + 2y + 3z &= 4 \\ 2x + 3y + 4z &= 7 \\ 3x + 4y + 5z &= 10 \end{aligned}$$

(i) Write down the coefficient matrix $\mathbb{A}$ and the augmented matrix $\mathbb{M}$.

(ii) Define the terms *rank*, *consistent* and *inconsistent*. Complete the following statements:

i. If rank($\mathbb{A}$) = rank($\mathbb{M}$) then ...

ii. If rank($\mathbb{A}$) < rank($\mathbb{M}$) then ...

(iii) Find a matrix in row reduced echelon form which is equivalent to the augmented matrix $\mathbb{M}$ and hence state rank($\mathbb{M}$) and rank($\mathbb{A}$) .

(iv) Find all solutions of the system if any exist.

(b) Use partial fractions to show that

$$\int_1^3 \frac{(3x+7)}{(x+1)(x+3)} = \log 6.$$

Question 3 [20 marks, 6+8+6]

Let

$$\mathbb{M} = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix}$$

be a matrix with entries in $\mathbb{R}$.

(a) What does it mean to say that v is an *eigenvector* of $\mathbb{M}$ with *eigenvalue* λ? State a condition for λ to be an eigenvalue of $\mathbb{M}$ in terms of the characteristic polynomial.

(b) Compute the characteristic polynomial of $\mathbb{M}$ and hence find the eigenvalues of $\mathbb{M}$. (You should find that $\mathbb{M}$ has three distinct eigenvalues.)

(c) For each eigenvalue λ, compute the eigenspace E_λ.

Question 4 [20 marks, 4+4+4+8]

(a) The matrices $\mathbb{A}$ and $\mathbb{B}$ are deined by

$$\mathbb{A} = \begin{pmatrix} \lambda & 2 \\ 4 & \lambda \end{pmatrix} \quad \text{and} \quad \mathbb{A} = \begin{pmatrix} \lambda & 2 \\ 4 & \lambda \end{pmatrix}.$$

Show that there is a value of λ for which $\mathbb{A} + \mathbb{B} + \mathbb{A}\mathbb{B} = \mu\mathbb{I}$, where μ is an integer and $\mathbb{I}$ is the 2×2 identity matrix, and state the corresponding value of λ.

(b) Find $\int x^2(x^2+1)\mathrm{d}x$.

(c) Show that $\int_0^9 (\sqrt{x} - 2)\, \mathrm{d}x = 0$. How do you interpret this in terms of areas?

(d) Let

$$\mathbb{A} = \begin{pmatrix} 5 & 1 & 3 & 8 \\ 3 & 2 & 2 & 5 \\ 1 & 0 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \vec{d} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}.$$

Find the general solution of $\mathbb{A}\vec{x} = \vec{d}$. (Put the augmented matrix into reduced row echelon form.) Express your solution in vector form.

Question 5 [20 marks, 4+5+6+5]

(a) Find the derivatives $\frac{dy}{dx}$ of the following functions

(i) $x^3 + y^3 + 3xy = 3$,

(ii) $y = x^{\frac{1}{x}}$

(b) Use Newton's method with five (5) iterations and five decimal places to find the root of

$$f(x) = x^3 - x + 1$$

given that $x_0 = -1$.

(c) Evaluate the following integral, $\int_0^2 \int_{-1}^1 (1 - 6x^2y)\, dx\, dy$.