UNIVERSITY OF SWAZILAND

RE-SIT EXAMINATION PAPER 2017/2018

TITLE OF PAPER	:	MATHEMATICS FOR STATISTICIANS
COURSE CODE	:	STA213
TIME ALLOWED	:	TWO (2) HOURS
REQUIREMENTS	:	CALCULATOR
INSTRUCTIONS	:	THIS PAPER HAS FIVE (5). ANSWER ANY THREE (3) QUESTIONS.

Question 1 [20 marks, 8+2+2+2+6]

(a) Suppose that a is a positive number and that the function f is given by

$$f(x,y) = x^4 - 2a^2x^2 - y^2 + 1.$$

Find the critical points of f. For each critical point of f, determine whether it is a local minimum, local maximum, or a saddle point.

(b) Let V be a vector space over a field $\mathfrak{F}$, and suppose that $\vec{v}_1, \cdots, \vec{v}_n$ are vectors in V.

(i) What does it mean to say that $\vec{v}_1, \cdots, \vec{v}_n$ are *linearly independent*?

(ii) What does it mean to say that $\{\vec{v}_1, \cdots, \vec{v}_n\}$ *spans* the vector space V ?

(iii) When is a set of vectors in V a basis?

(iv) Let $V = \mathbb{R}^3$. Prove that the set

$$\mathfrak{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}$$

is a basis for V.

Question 2 [20 marks, 8+8+4]

(a) The number of fish in a lake is 10000 at the start of 2016. Each year, 5% of the fish who were alive at the start of the year die and 1000 new fish are born. Find an expression, in as simple a form as possible, for the number of fish in the lake N years after the start of 2016. Describe what happens to the number of fish in the lake in the long run.

(b) Find all eigenvalues and eigenvectors of the matrix

$$\mathbb{A} = \begin{pmatrix} 1 & 2 \\ 0 & 6 \end{pmatrix}$$

(c) Find a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with non-zero entries in $\mathbb{R}$ such that the equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

has no solutions.

Question 3 [20 marks, 4+6+4+2+4]

(a) Let the following matrix and vector,

$$\mathbb{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

be given.

(i) Determine the matrix $\mathbb{B} = \mathbb{A}^T\mathbb{A}$ as well as its inverse and state the formula for the projection matrix that projects $\vec{b}$ onto the column space of $\mathbb{A}$.

(ii) Determine the projection vector $\vec{p} = \mathbb{A}\left(\mathbb{A}^T\mathbb{A}\right)^{-1}\mathbb{A}^T\vec{b}$ for projecting $\vec{b}$ onto the column space of $\mathbb{A}$ as well as the vector $\vec{e}_\perp$ that is orthogonal to the column space of $\mathbb{A}$ such that $\vec{b} = \vec{p} + \vec{e}_\perp$.

(b) Find a value of $\alpha \in \mathbb{C}$ for which the vectors

$$\vec{w}_1 = \begin{pmatrix} 1 \\ 0 \\ i \\ 2 \end{pmatrix} \quad \text{and} \quad \vec{w}_2 = \begin{pmatrix} 01 \\ 3 \\ \alpha \\ 1-i \end{pmatrix}$$

are orthogonal in the standard inner product on $\mathbb{C}^4$.

(c) Consider the 4×2 matrix

$$\mathbb{A} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Verify that $\mathbb{A}$ has rank 2

(d) Use Newton's method with five (5) iterations and five decimal places to find the root of

$$f(x) = x^3 - x + 1$$

given that $x_0 = -1$.

Question 4 [20 marks, 1+1+1+5+12]

(a) Consider the following system of linear equations:

$$\begin{aligned} x + 0y + z &= 1 \\ x + y + z &= 2 \\ 0x + y + z &= 3 \end{aligned}$$

(i) Compute the coefficient matrix. Call that matrix $\mathbb{A}$

(ii) Compute the augmented matrix for the given system of linear equations.

(iii) Is the matrix $\mathbb{A}$ in row echelon form? (Justify your answer)

(iv) Use Gauss-Jordan elimination for solving a system of linear equations to solve the given system of linear equation

(b) Suppose the quantity y is defined as a function of x through the equation

$$x^2y^5 - 6x^3y^2 + 2xy = 1.$$

Find the value of y when $x = 1/2$. [You might find it useful to note that $y^3 3y^2 + 4y 4 = (y 2)(y^2 y + 2)$.] Find a general expression for the derivative $y'(x) = \frac{dy}{dx}$. Hence determine the value of $y'(1/2)$.

Question 5 [20 marks, 6+6+4+4]

(a) A monopoly has fixed costs of 20 and marginal cost function $3q^2 + 4$. The demand equation for its product is $p + q = 20$. Determine the profit function in terms of q. Hence find the production level that maximises the profit.

(b) Use the method of Lagrange multipliers to find the positive values of x and y which maximise

$$\frac{2y}{y+2} + \frac{x}{x+1}$$

subject to the constraint $x + y = 120$.

(c) The matrix $\mathbb{C}$ is defined by

$$\mathbb{C} = \begin{pmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{pmatrix}.$$

Factorize $\det(\mathbb{C})$ as the product of three linear factors.

(d) The matrix $\mathbb{A}$ is defined by $\mathbb{A} = \begin{pmatrix} -2 & c \\ d & 3 \end{pmatrix}$, where c and d are constants. Given that the image of the point $(5, 2)$ under the transformation represented by matrix $\mathbb{A}$ is $(-2, 1)$, find the values of c and d.