UNIVERSITY OF SWAZILAND

FINAL EXAMINATION PAPER 2017

TITLE OF PAPER	:	INTRODUCTION TO STOCHASTIC PROCESSES
COURSE CODE	:	STA303
TIME ALLOWED	:	TWO (2) HOURS
REQUIREMENTS	:	CALCULATOR
INSTRUCTIONS	:	THIS PAPER HAS FIVE (5) QUESTIONS. ANSWER ANY THREE (3) QUESTIONS.

Question 1 [20 marks, 6+2+4+8]

(a) A markov chain Z_t, $t = 0, 1, 2, \cdots$ on the state space $\mathfrak{S} = \{1, 2, 3, 4, 5\}$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0.3 & 0.4 & 0.3 & 0 & 0 \\ 0 & 0 & 0.3 & 0.4 & 0.3 \\ 0 & 0.3 & 0 & 0.3 & 0.4 \\ 0 & 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0 & 0.6 & 0.4 \end{pmatrix} \end{array}.$$

Classify all states. Which states are recurrent and which states are transient? Find all closed classes. Is the chain *ergodic*?

(b) Customers arrive at a bank according to a Poisson process with rate 6 customers per hour.

(i) What is the probability that five customers arrive in the first hour after the bank opens?

(ii) What is the probability that five customers arrive in the first hour after the bank opens and that three customers arrive in the second hour after the bank opens?

(iii) Suppose it is known that five customers arrived in the first hour after the bank opened. What is the probability that exactly one of the five customers arrived during the first 20 minutes? Justify your answer using a theorem.

Question 2 [20 marks, 2+4+3+3+8]

(a) State what is meant by the memoryless property of the exponential distribution.

(b) Prove the memoryless property for the exponential distribution.

(c) Let $\mathbf{G}$ be the generator of a continuous time Markov chain $X(t)$, with $t \geq 0$, and let $\mathbf{P}(t)$ be the matrix such that $\mathbb{P}(t)_{i,j} = \mathbb{P}(X(s+t) = j | X(s) = i)$.

(i) State the equation for $\mathbf{P}(t)$ in terms of $\mathbf{G}$.

(ii) State the backwards and forwards Kolmogorov equations.

(d) A markov chain Z_t, $t = 0, 1, 2, \cdots$ on the state space $\mathfrak{S} = \{0, 1, 2, 3, 4\}$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Determine the mean time to reach state 4 starting from state 1 using a first step analysis.

Question 3 [20 marks, 8+6+6]

(a) A markov chain Z_t, $t = 0, 1, 2, \cdots$ on the state space $\mathfrak{S} = \{1, 2, 3, 4\}$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0.2 & 0.2 & 0.3 & 0.3 \\ 0.5 & 0 & 0.5 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \end{pmatrix} \end{array}$$

with the three-step transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0.383 & 0.243 & 0.287 & 0.087 \\ 0.320 & 0.220 & 0.355 & 0.105 \\ 0.320 & 0.355 & 0.230 & 0.105 \\ 0.350 & 0.225 & 0.275 & 0.150 \end{pmatrix} \end{array}.$$

If the initial probability distribution is

$$\{0.3, 0.3, 0.2, 0.2\}$$

what is $\mathbb{P}(Z_3 = 2, Z_6 = 3, Z_9 = 4)$?

(b) A small barbershop is operated by two barbers, each of which has a chair for a single customer. When two customers are in the store, its doors are locked so no more customers can enter. The remaining time, when there are fewer than two customers in the store, its doors are left open and potential customers arrive according to a Poisson process with rate of three per hour. Successive service times are independent exponential random variables with expectation of $\frac{1}{4}$ hour.

(i) Find the generator for $X(t)$, the number of customers in the shop.

(ii) Find the limiting probabilies $\pi_i = \lim_{t\to\infty} \mathbb{P}(X(t) = i)$ for $i = 0, 1, 2$.

Question 4 [20 marks, 6+6+8]

(a) A markov chain Z_t, $t = 0, 1, 2, \cdots$ on the state space $\mathfrak{S} = \{1, 2, 3\}$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.5 & 0 & 0.5 \\ 0.6 & 0 & 0.4 \\ 0.4 & 0.6 & 0 \end{pmatrix} \end{array}.$$

Find the stationary probability distribution.

(b) A person sends a chain letter which takes exactly six days to arrive. At the start of each week, each person who has just received the letter sends it to 0, 1 and 2 people with probabilities 0.3, 0.4 and 0.3 respectively. Suppose that X_n denotes the number of individuals who receive the letter on the sixth day of the nth week. Let μ_n be the mean value of X_n and $\mu = \mu_1$. Let $G_n(s)$ be the probability generating function of X_n and $G(s) = G_1(s)$ (so $X_0 = 1$). What is the probability that the chain ultimately dies out? [You may use any general result from the theory of branching processes provided you state it clearly]

(c) Let Z_t, $t = 0, 1, 2, \cdots$ be a Markov chain on a finite state space $\mathfrak{S} = \{1, 2, \cdots, n\}$. State the Markov property clearly. Define the 3-step transition probabilities

$$p_{ij}^{(3)} = \mathbb{P}(Z_{t+3} = j | Z_t = i) \qquad i, j \in \mathfrak{S}, \quad t = 0, 1, 2, \cdots.$$

Using the Markov property, show

$$\mathbb{P}(Z_6 = j | Z_0 = i) = \sum_{k=1}^{n} p_{ik}^{(3)} p_{kj}^{(3)}$$

where $i, j \in \mathfrak{S}$.

Question 5 [20 marks, 6+6+8]

(a) Suppose a machine can be in one of three states: Good, Fair or Bad. After entering state Good, the machine remains there for a random time having expectation of one year, after which it enters state Fair; after entering state Fair, the machine remains there for a random time having expectation of six months, after which it enters state Bad; after entering state Bad, the machine remains there a random time having expectation of six months, after which it either enters state Good with probability one half or returns to state Bad with probability one half.

(i) In the long run, what is the proportion of transitions to states Good, Fair and Bad, respectively?

(ii) In the long run, what is the proportion of time the machine is in the states Good, Fair and Bad, respectively?

(b) A gambler has SZL 5 and has the the opportunity of playin in a game in which the probability is 0.3 that he wins an amount equal to his stake, and probability 0.7 that he loses his stake. If his capital is increased to SZL 8, he will stop playing. If his capital becomes 0, he has to stop of course. He is allowed to decide how much to state at each game, and he decides each time to bet , when this is possible, just sufficint in order to increase his capital to SZL 8, or if he does not have enough capital (in Emalangeni) after the nth game. Then Z_n is a Markov chain. Identify the state space. Find the transition probability matrix $\mathbf{P}$. Compute the probability of ultimately increasing his capital to SZL 8 using first step analysis.