UNIVERSITY OF SWAZILAND

RE-SIT EXAMINATION PAPER 2017/2018

TITLE OF PAPER	:	INTRODUCTION TO STOCHASTIC PROCESSES
COURSE CODE	:	STA303
TIME ALLOWED	:	TWO (2) HOURS
REQUIREMENTS	:	CALCULATOR
INSTRUCTIONS	:	THIS PAPER HAS FIVE (5) QUESTIONS. ANSWER ANY THREE (3) QUESTIONS.

Question 1 [20 marks, 4+6+10]

(a) Components in a machine fail and are replaced according to a Poisson process of rate 4 a month.

 (i) Find the probability that exactly 4 fail in the first month and exactly 8 fail in the first two months.

 (ii) Find the probability that at least 2 fail in the first month and at least 4 fail in the first 2 months.

(b) A population starts off with just one female. Each female lives for a fixed time T and then dies. The number of offspring of a single female is given by the following probability distribution:

m	0	1	2
p_m	0.3	0.4	0.3

Different female are independent. Let $G_n(s)$ denote the probability generating function of the number of females at the nth generation and write $G_1(s) = G(s)$. Write down the polynomial expression for $G(s)$. Working from first principles use a first step argument to show that

$$G_2(s) = G(G(s)).$$

Hence deduce the probability distribution of the females in the second generation.

Question 2 [20 marks, 6+8+6]

(a) A markov chain Z_t, $t = 0, 1, 2, \cdots$ on the state space $\mathfrak{S} = \{1, 2, 3, 4, 5\}$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0 & 0.3 & 0.3 & 0.2 & 0.2 \\ 0 & 0 & 0.3 & 0.3 & 0.4 \\ 0 & 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0 & 0.6 & 0.4 \end{pmatrix} \end{array}$$

with the two-step transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0.04 & 0.10 & 0.1*6 & 0.34 & 0.36 \\ 0 & 0.09 & 0.18 & 0.35 & 0.38 \\ 0 & 0 & 0.09 & 0.45 & 0.46 \\ 0 & 0 & 0 & 0.52 & 0.48 \\ 0 & 0 & 0 & 0.48 & 0.52 \end{pmatrix} \end{array}.$$

 (i) If the initial probability distribution is

$$\{0.2, 0.2, 0.2, 0.3, 0.1\}$$

 what is $\mathbb{P}(Z_2 = 4 \text{ or } 5)$?

(ii) Determine the mean time to reach state 4 or 5 starting from state 1 using a first step analysis.

(b) Let Z_t, $t = 0, 1, 2, \cdots$ be a Markov chain on a finite state space $\mathfrak{S} = \{1, 2, \cdots, n\}$. State the Markov property clearly. Define the 1-step transition probabilities

$$p_{ij} = \mathbb{P}(Z_{t+1} = j | Z_t = i) \qquad i, j \in \mathfrak{S}, \quad t = 0, 1, 2, \cdots .$$

Using the Markov property, show

$$\mathbb{P}(Z_2 = j | Z_0 = i) = \sum_{k=1}^{n} p_{ik} p_{kj}$$

where $i, j \in \mathfrak{S}$.

Question 3 **[20 marks, 8+2+4+6]**

(a) A markov chain Z_t, $t = 0, 1, 2, \cdots$ on the state space $\mathfrak{S} = \{1, 2, 3\}$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.6 & 0 & 0.4 \\ 0.3 & 0.7 & 0 \end{pmatrix} \end{array}.$$

Explain why the stationary distribution exists. Find the stationary probability distribution.

(b) A markov chain Z_t, $t = 0, 1, 2, \cdots$ on the state space $\mathfrak{S} = \{1, 2, 3, 4, 5\}$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 & 0 \\ 0.7 & 0 & 0 & 0.3 & 0 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

(i) Which states are absorbing?

(ii) Find the probability of absorption occurs at state 5 starting from state 3. (You need to set up the difference equations by first step analysis and solve them.)

(c) Let Z_t, $t = 0, 1, 2, \cdots$ be a Markov chain on a finite state space $\mathfrak{S} = \{1, 2, \cdots, n\}$. State the Markov property clearly. Define the 1-step transition probabilities

$$p_{ij} = \mathbb{P}(Z_{t+1} = j | Z_t = i) \qquad i, j \in \mathfrak{S}, \quad t = 0, 1, 2, \cdots .$$

Using the Markov property, show

$$\mathbb{P}(Z_2 = j | Z_0 = i) = \sum_{k=1}^{n} p_{ik} p_{kj}$$

where $i, j \in \mathfrak{S}$.

Question 4 [20 marks, 3+9+6+2]

(a) Let $X(t)$ be a continuous time Markov chain with conditional probability densities

$$f_n(y_n, t_n | y_{n-1}, t_{n-1}; y_{n-1}, t_{n-2}; \cdots ; y_1, t_1)$$

where $0 \leq t_1 < t_2 < \cdots t_m$ and $y_i \in \mathbb{R}$ for all $1 \leq i \leq n$, where $\mathbb{R}$ is the set of real numbers. State what is meant by the Markov property for $X(t)$.

(b) A system consists of two machines and one repairman. All machines in operating condition are operated. The amount of time in hours that an operating machine works before breaking down is exponentially distributed with mean 20. The amount of time in hours that it takes the repairman to fix a machine is exponentially distributed with mean 1. The repairman can work on only one failed machine at any given time. Let $X(t)$ be the number of machines in operating condition at time t.

(i) Find the generator for $X(t)$.

(ii) Calculate the long run probability distribution for X(t).

(iii) In the long run, what is the proportion of time that the repairman is fixing a machine?

Question 5 [20 marks, 3+3+7+7]

(a) Let $\mathbf{G}$ be the generator of a continuous time Markov chain $X(t)$, with $t \geq 0$, and let $\mathbf{P}(t)$ be the matrix such that $\mathbb{P}(t)_{i,j} = \mathbb{P}(X(s+t) = j | X(s) = i)$.

(i) State the equation for $\mathbf{P}(t)$ in terms of $\mathbf{G}$.

(ii) State the backwards and forwards Kolmogorov equations.

(b) Consider a fleet of three buses. Each bus breaks down independently at rate μ, after which it is sent to the depot for repairs. The repair shop can repair up to two buses at a time and each bus takes an exponential amount of time with parameter λ to repair.

(i) Find the generator for $X(t)$, the number of buses in service.

(ii) Find the limiting probability

$$\pi_0 = \lim_{t \to \infty} \mathbb{P}(X(t) = 0).$$