UNIVERSITY OF ESWATINI

SUPPLEMENTARY EXAMINATION PAPER 2018/2019

TITLE OF PAPER : INTRODUCTION TO STOCHASTIC PROCESSES

COURSE CODE : STA303

TIME ALLOWED : TWO (2) HOURS

REQUIREMENTS : CALCULATOR

INSTRUCTIONS : ANSWER ANY THREE QUESTIONS

Question 1 [20 marks, 4+6+4+6]

(a) The joint density of (X, Y) is

$$f_{X,Y}(x,y) = \begin{cases} x^k e^{-ky}, & 0 < x < y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

(i) Verify that this is a valid density when $k = 1$.

(ii) Find $f_X(x)$, $f_Y(y)$ and $f_{Y|X}(y|x)$, representing the marginal density of X, Y and the conditional density of Y given $X = x$ respectively.

(b) Let $X \sim$ Bernuolli(p), that is, $P(X = x) = p^x(1-p)^{1-x}$, $\quad x = 0, 1$. Find the moment generating function of X. Let $Y = NX$, when N Poisson(μ) with

$$P(N = n) = \frac{\mu^n e^{-\mu}}{n!} \quad n = 0, 1, \cdots,$$

and N is independent of X. Derive the moment generating function of Y.

Question 2 [20 marks, 6+5+9]

Suppose you apply for a passport. If you do not receive further notice from the related government department (with probability $1/2$), the waiting time T_1 until you receive your new passport is exponentially distributed with parameter λ, that is $T_1 \sim \exp(\lambda)$, with probability density function

$$f_{T_1}(t) = \lambda \exp(-\lambda t), \quad t > 0.$$

However, there is a probability of $1/3$ that there is a minor delay in processing your application, and the waiting time T_2 until you receive your new passport will then be $T_2 \sim \exp(\lambda/2)$.

There is also a probability of $1/6$ that there is a serious delay. The waiting time T_3 until you receive your new passport will then be $T_3 \sim \exp(\lambda/4)$.

All time units are in months.

(a) Let T be the overall waiting time until you receive the new passport. Find $\mathbb{P}(T < 4)$ in terms of λ.

(b) You have waited for 4 months and still have not received your passport. What is the probability (in terms of λ) that there are actually delays (minor or serious) in your application?

(c) Suppose $\lambda = 1$. Find the mean and variance of T. (You can use the mean and variance of an exponential random variable without proof, as long as you state them clearly. Hint: For the variance of T, find $E(T^2)$ first.)

Question 3 [20 marks, 4+3+3+3+7]

In the Ultrastratos civilisation on planet Anachronista, the population is divided into four strata which, in order of status, are labelled Alpha, Beta, Gamma and Delta. By the traditions of the civilisation, no child can have a status more than one different from its parents. As examples, in each generation 20% of the children of Alphas grow up to be Betas, the rest remaining Alphas, while of the Beta offspring 50% remain Betas while 10% become Alphas and the rest Gammas.

A Markov chain describes the status of members of the population at successive generations. Its transition matrix is given by $\mathbb{P}$, defined below, in which some entries labelled $*$ have been omitted.

$$\mathbb{P} = \begin{matrix} Alpha \\ Beta \\ Gamma \\ Delta \end{matrix} \begin{pmatrix} * & 0.2 & * & 0 \\ 0.1 & 0.5 & * & * \\ * & * & 0.5 & 0.4 \\ * & 0 & * & 0.8 \end{pmatrix}$$

(a) Use the information above to fill in the missing elements of $\mathbb{P}$ and calculate the two-step-ahead transition matrix.

(b) What is the probability that the child of a Beta becomes a Gamma? Show that the grandchild of a Gamma is twice as likely to become a Delta as the grandchild of a Delta is to become a Gamma.

(c) Calculate the probability that, after four generations, a descendant of an Alpha is a Delta.

(d) Explain how you can tell that the chain consists of a single irreducible class. Are any of the states transient? Give your reason.

(e) Find the stationary distribution of the chain. If the population initially has no Deltas, what will be the proportion of Deltas after a large number of generations?

Question 4 [20 marks, 5+5+5+5]

(a) Let $N(t)$ be the number of telephone calls at an exchange in the interval $(0, t]$ We suppose that $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda = 10$ per hour. Calculate the probability that no calls will be received during each of two consecutive 15-minute periods.

(b) Let $N(t)$ be the number of failures in the interval $(0, t]$. We suppose that $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda = 1$ per week. Calculate the probability that

(i) the system operates without failure during two consecutive weeks,

(ii) the system will have exactly two failures during a given week, knowing that it operated without failure during the previous two weeks,

(iii) less than two weeks elapse before the third failure occurs

Question 5 [20 marks, 6+6+8]

(a) The annual number of hurricanes forming in the Atlantic basin has a Poisson distribution with parameter λ. Each hurricane that forms has probability p of making landfall independent of all other hurricanes. Let X be the number of hurricanes that form in the basin and Y be the number that make landfall. Find:

(i) $E(Y)$,

(ii) $\text{Corr}(X, Y)$.

(b) Let X be such that the distribution of X given $Y = y$ is Poisson, parameter y. Let $Y \sim$ Poisson(μ). Show that

$$G_{X+Y}(s) = \exp\left\{\mu(se^{s-1} - 1)\right\}$$